Light-Cone Quantization of Electrodynamics[1]

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arXiv:hep-th/9410240v1 31 Oct 1994

Abstract

Light-cone quantization of (3+1)-dimensional electrodynamics is discussed, using discretization as an infrared regulator and paying careful attention to the interplay between gauge choice and boundary conditions. In the zero longitudinal momentum sector of the theory a general gauge fixing is performed and the corresponding relations that determine the constrained modes of the gauge field are obtained. The constraints are solved perturbatively and the structure of the theory is studied to lowest nontrivial order.

1. Introduction

The primary motivation for considering light-cone quantization when attempting to derive a constituent picture of hadrons from QCD is that a single cutoff, removing states of small longitudinal momentum, renders the vacuum of the theory completely trivial. This immediately results in a picture in which all partons in a physical hadronic state are tied to the hadron, instead of being disconnected excitations in a complicated vacuum. The situation is quite different when quantizing on a spacelike surface, where no simple cutoff can remove the states that are kinematically allowed to mix with the bare Fock vacuum. In this case a constituent description would necessarily be in terms of quasiparticle states, i.e., collective excitations above a complicated ground state.

One calculational scheme which accomplishes (in most cases) this small-k^+ cutoff is "discretized" light-cone quantization (DLCQ). In this approach one imposes periodicity conditions on the fields in the x^- direction, leading to a discrete set of allowed longitudinal momenta. For Fermi fields it is permissible to choose antiperiodic boundary conditions so that there is no zero-momentum mode. For bosonic fields one must generally choose periodic boundary conditions, but the zero mode is in most cases not an independent field; rather, it is a constrained functional of the other, dynamical fields in the theory.

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[1] Talk presented at "Theory of Hadrons and Light-Front QCD," Polana Zgorzelisko, Poland, August 1994.

The bare vacuum is thus the only state in the theory with $k^+ = 0$, and so is an exact eigenstate of the full Hamiltonian. The price we pay for this simplification is that we must solve the constraints that determine the zero modes.

The exception to this is gauge theories. Here one finds that the zero modes of A^i and A^- are constrained fields, but certain of the zero modes of A^+ can in fact be dynamical. Furthermore, these cannot be removed from the theory by a gauge choice. Thus in this case there are $k^+ = 0$ particle states that can in principle mix with the bare vacuum to give a nontrivial physical vacuum state, although there are many fewer such states than typically occur in equal-time quantization. In this case the nontrivial vacuum must either be confronted or further *ad hoc* truncations must be made in order to obtain a trivial vacuum.

In this talk I summarize recent work with A. Kalloniatis on the DLCQ of Abelian gauge theory with fermions [1]. The central problem is disentangling the constrained and dynamical zero modes in the context of a particular gauge fixing of the theory. After this has been done, we then face the problem of solving the relations that determine the constrained zero modes so that the Hamiltonian can be written down. I shall describe here the formulation of the theory to lowest order in perturbation theory.

2. Gauge Fixing and the Zero Momentum Modes

In DLCQ the theory is defined in a light-cone box, with $-L_\perp \leq x^i \leq L_\perp$ and $-L \leq x^- \leq L$, and with some boundary conditions imposed on the fields. Because the gauge field couples to a fermion bilinear, which is necessarily periodic in all coordinates, A_μ must be taken to be periodic in both x^- and $x_\perp$. We have more flexibility with the Fermi field, and it is convenient to choose this to be periodic in $x_\perp$ and antiperiodic in x^-. This eliminates the zero longitudinal momentum mode while still allowing an expansion of the field in a complete set of basis functions.

These functions are taken to be plane waves, and for periodic fields there will of course be zero-momentum modes. In previous work on the formulation of QED in DLCQ [2] the zero modes were discarded, and only the nonzero, or "normal," modes were retained. This part of the theory is essentially unchanged. The "global" zero modes—those independent of all spatial coordinates—require special treatment, and the reader is referred to Appendix B of ref. [1] for more details. (It is shown there that they are irrelevant for the calculations we shall present here.) The quantities that will be of central interest to us here are the "proper" zero modes, which are independent of x^- but *not* independent of $x_\perp$. Specifically, for a periodic quantity f, the proper zero mode is defined as

$$f_0(x_\perp) \equiv \int_{-L}^{L} \frac{dx^-}{2L} f(x^-, x_\perp) - \int_{-L}^{L} \frac{dx^-}{2L} \int_{-L_\perp}^{L_\perp} \frac{d^2 x_\perp}{(2L_\perp)^2} f(x^-, x_\perp) \, . \tag{1}$$

Projecting the equation of motion $\partial_\mu F^{\mu\nu} = gJ^\nu$ onto the proper zero mode sector gives

$$- \partial_\perp^2 A_0^+ = gJ_0^+ \,, \tag{2}$$

$$- 2(\partial_+)^2 A_0^+ - \partial_\perp^2 A_0^- - 2\partial_i\partial_+ A_0^i = gJ_0^- \,, \tag{3}$$

$$- \partial_\perp^2 A_0^i + \partial_i\partial_+ A_0^+ + \partial_i\partial_j A_0^j = gJ_0^i \,. \tag{4}$$

We first observe that eq. (2) is a constraint which determines A_0^+ in terms of J^+. Eqs. (3) and (4) then determine A_0^- and A_0^i. Thus all of the proper zero modes are constrained fields.

Eq. (2) is clearly incompatible with the light-cone gauge $A^+ = 0$, which is most natural in light-cone analyses of gauge theories. Here we encounter a common problem in treating axial gauges on compact spaces, which has nothing to do with light-cone quantization *per se*. The point is that the x^--independent part of A^+ is in fact gauge invariant, since under a gauge transformation

$$A^+ \to A^+ + 2\partial_-\omega \,, \tag{5}$$

where ω is a function periodic in all coordinates. Thus it is not possible to bring an arbitrary gauge field configuration to one satisfying $A^+ = 0$ via a gauge transformation. We can (and will) set the normal mode part of A^+ to zero, which is equivalent to

$$\partial_- A^+ = 0 \,. \tag{6}$$

This does not, however, completely fix the gauge—we are free to make arbitrary x^--independent gauge transformations without undoing eq. (6). We may therefore impose further conditions on A_μ in the zero mode sector of the theory.

To see what might be useful in this regard, let us consider solving eq. (4) for the transverse zero modes. By making use of current conservation we can rewrite eq. (4) as

$$- \partial_\perp^2 \Big(\delta_j^i - \frac{\partial_i\partial_j}{\partial_\perp^2}\Big) A_0^j = g\Big(\delta_j^i - \frac{\partial_i\partial_j}{\partial_\perp^2}\Big) J_0^j \,, \tag{7}$$

the general solution of which is

$$A_0^i = -g\frac{1}{\partial_\perp^2} J_0^i + \partial_i\varphi(x^+, x_\perp) \,. \tag{8}$$

Here φ must be independent of x^- but is otherwise arbitrary, and reflects the residual gauge invariance. Imposing a condition on, say, $\partial_i A_0^i$ will uniquely determine this function.

Now different choices for φ merely correspond to different gauge choices in the zero mode sector, so we expect that physical quantities should be independent of the specific φ we choose. This is in fact the case. It turns out, however, that the unique choice which allows simple (free-field) commutation relations among the fields is $\varphi = 0$. The easiest

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way to see this is to note that for $\varphi \neq 0$ the kinematical operators P^i do not have their free-field forms [1]. Thus it would be impossible to realize the Heisenberg relation

$$[\psi_+, P^i] = -i\partial_i \psi_+ \tag{9}$$

with the usual anticommutation relation between ψ_+ and $\psi_+^\dagger$. The complicated anticommutator required could in principle be determined, but this is not really necessary. It is far simpler to choose the gauge $\varphi = 0$ and take the usual canonical commutation relations among the fields, and this is what we shall do.

3. Perturbative Formulation

We shall now construct a perturbative solution of the constraints and study the structure of the theory to lowest nontrivial order. This requires constructing the Hamiltonian through terms of $\mathcal{O}(g^2)$, which in turn corresponds to taking the $\mathcal{O}(g)$ solutions for the zero modes A_0^i and A_0^+. We obtain A_0^i at this order simply by setting $g = 0$ in the current J_0^i. A_0^+ is given exactly by inverting the $\partial_\perp^2$ in eq. (2).

With A_0^i and A_0^+ in hand it is simple to construct the Hamiltonian. The contributions from the normal mode part of the theory may be found in ref. [2].[2] To this we must add the contributions from the proper zero modes, which reduce to [1]

$$P_Z^- = \frac{g^2}{2} \int dx^- d^2 x_\perp \left(\frac{\partial_i}{\partial_\perp^2} J_0^\mu \right) \left(\frac{\partial_i}{\partial_\perp^2} J_{0\mu} \right) , \tag{10}$$

where the currents are evaluated with $g = 0$. It is a straightforward (if tedious) exercise to express P_Z^- in the Fock representation. We obtain various four-fermion operators, as well as fermion bilinears ("self-induced inertias") which arise when the four-point terms are brought into normal order.

We can now compute with this Hamiltonian and study the effects of the zero mode-induced interactions (10). We shall here describe the calculation of the fermion self-energy (eigenvalue of P^-) at $\mathcal{O}(g^2)$. This is not the only quantity to which P_Z^- contributes, of course. The four-fermion operators in P_Z^- will give rise to divergent contributions to the $e^+ e^- \gamma$ vertex, and hence to the charge renormalization at this order. They will also contribute to tree-level scattering amplitudes, *etc.*

In fact, we shall consider only the part of the self-energy that is quadratically divergent in a transverse momentum cutoff. The contributions coming from the normal mode sector of the theory may be taken from the work of Tang, *et al.* [2] (subject to the caveat in

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[2] Note that the Hamiltonian given in ref. [2] should be symmetrized under charge conjugation ($b \leftrightarrow d$). This is the effect of using the explicitly C-odd form of the current $J^\mu = \frac{1}{2}[\overline{\psi}, \gamma^\mu \psi]$, or equivalently of properly symmetrizing products of noncommuting operators in the construction of P^-.

footnote 2). It has the form

$$\delta P^- = c\Lambda_\perp^2 \left[\frac{2\ln 2}{p^+} - \frac{1}{(p^+)^2 L} \right] , \qquad (11)$$

where p^+ is the momentum of the fermion and c is a dimensionless constant.

Two points should be made about this result. The first is that it represents a failure of chiral symmetry. Corrections to the fermion mass are supposed to be proportional to m, so that they vanish in the chiral limit. Here, however, there is a correction to m [the first term in eq. (11)] that is completely independent of m. Furthermore, the second term in eq. (11) does not have the correct form to be interpreted as the redefinition of a parameter in the Lagrangian. It would have to be removed by a noncovariant counterterm.

There is another quadratically divergent contribution to the self-energy, however, coming from the zero mode-induced interactions (10). It reduces to

$$\delta P_{zm}^- = +c\Lambda_\perp^2 \left[\frac{1}{(p^+)^2 L} \right] , \qquad (12)$$

so that the noncovariant part of the quadratic divergence is in fact canceled when the zero modes are retained. Thus inclusion of the zero modes renders the UV behavior of the theory more benign. The quadratic divergence proportional to $\frac{1}{p^+}$ survives, unlike in a continuum formulation (with a Lorentz-covariant regulator), but this may be removed by a redefinition of the fermion kinetic mass.

4. Discussion

The box length L is introduced to regulate the theory in the infrared, and in principle it should be taken to infinity at the end of the day. More concretely, one should calculate physical quantities of interest, which may be functions of the parameter L, and study them in the limit $L \to \infty$. For the energy of the one-fermion state, for example, this means taking $L \to \infty$ with the p^+ of the state held fixed. Thus the term in eq. (11) that is canceled by the zero mode contribution is explicitly L-dependent, in addition to being quadratically divergent in $\Lambda_\perp$. It is in some sense "irrelevant"—it goes to zero as the infrared cutoff is removed—but this is not the point we wish to emphasize. Rather it is that the zero mode-induced interactions *removed* this L-dependence. This suggests that at least some of the zero mode interactions can be thought of as infrared counterterms, that serve to remove dependence on the infrared cutoff L.

In a theory like QED, in which we do not expect any vacuum structure (in the sense of, e.g., vacuum expectation values or a θ-vacuum), we therefore expect that inclusion of the zero mode interactions mainly helps to remove L-dependence from the solution of the theory and so makes the infinite volume limit easier to reach. There is numerical evidence

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that this is the case [3]. A useful analogy might be with improved actions for lattice theories, in which dependence on the lattice spacing a is explicitly removed through some order. Calculations using the improved action can then be done at a larger value of a (fewer lattice points for a given physical volume) for a fixed numerical accuracy.

It is clearly of interest to learn whether there are zero mode interactions that are marginal or relevant in the renormalization group sense. This question can perhaps be addressed systematically from the point of view of the light-cone power-counting analysis of Wilson [4].

It is also important to develop more sophisticated, nonperturbative methods for solving the constraints. For QED with a realistic value of the electron charge, however, it is possible that a perturbative treatment of the constraints could suffice; that is, that we could use a perturbation theory to construct the Hamiltonian, and then diagonalize it nonperturbatively. This approach is similar in spirit to that advocated in ref. [4], where the idea is to use a perturbative realization of the renormalization group to construct an effective Hamiltonian for QCD, which is then solved nonperturbatively.

We have not addressed here the general issue of removing dependence on the transverse cutoff $\Lambda_\perp$ from the theory. This is perhaps the more difficult problem, and in fact it may make its presence felt in calculations like those presented here. The constraint relations are strictly speaking ill-defined due to transverse UV divergences, and how one regulates and removes these presumably affects the treatment of the zero modes. This may be an important issue for theories with nontrivial UV divergences.

Acknowledgements

References

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